Exact Bianchi IV cosmological model*

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An analytical global solution to the Einstein field equations for the case of Bianchi IV symmetry is constructed. It is axisymmetric, has a singularity in the finite past, is asymptotically flat as $t \to \infty$, does not admit the presence of a perfect fluid source, and must have zero cosmological constant.

INTRODUCTION

In recent years there has been strong interest in spatially homogeneous, nonisotropic, cosmological models based on the Einstein theory of general relativity. These are the so-called Bianchi models. Extensive discussions have been published by Ellis and MacCallum,[1] Ryan and Shepley,[2] MacCallum,[3] and others. Among the reasons for the interest in such models are the following: the source-free models generally have an anti-Machian character[4] manifested by nonzero curvature, a fact first noted by Taub[5] and yet to be satisfactorily explained; the evidence for the isotropic nature of the universe although strong for the present epoch (see, e.g., Peebles[6]) is less convincing for earlier regimes.

The particular case studied here is a Bianchi IV model. We find that for the simplest choice of metric a solution in closed form is obtainable. It is axisymmetric, has a singularity in the finite past, and is asymptotically flat as $t \to \infty$. The model has a number of remarkable features: It is first found that it cannot be persistently diagonal in either the vacuum or perfect-fluid situation; the simplest nondiagonal case will possess a solution only if the cosmological constant is zero and no matter is present. By "matter" is meant a perfect fluid. There is some evidence that an electromagnetic field may be admissible as a source. For the most part we use Cartan symbolic[2] methods to obtain components of the Ricci tensor. Although many authors have derived general expressions for the components of this tensor, it proved to be advantageous to treat the Bianchi IV model in its essential specialized form from the start.

FIELD EQUATIONS

The Einstein field equations with cosmological constant are

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = kT_{\mu\nu} , \quad (1)$$

with $k = 8\pi G/c^4$ and units of G and c so chosen that $k = 1$. Greek indices run from 0 to 3; Latin indices from 1 to 3. The metric signature is +2. The summation convention is used throughout. The stress-energy tensor for a perfect fluid is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} , \quad (2)$$

where ρ is the energy density, p is the pressure, and u_μ is the fluid velocity with $u^\mu u_\mu = -1$. The field equations may be rewritten in the form

$$R_{\mu\nu} = (\rho + p)u_\mu u_\nu + \tfrac{1}{2}g_{\mu\nu}(\rho - p) + g_{\mu\nu}\Lambda. \quad (3)$$

The line element in a synchronous system in terms of one-forms is

$$ds^2 = -\sigma^0\sigma^0 + g_{ij}(t)\sigma^i\sigma^j . \quad (4)$$

It is a characteristic feature of the Bianchi models that a synchronous metric in which the spatial portion of the metric is parametrized by t is generally admissible.

The exterior differentials of the one-forms for Bianchi IV symmetry are

$$d\sigma^1 = 0, \quad (5a)$$

$$d\sigma^2 = \sigma^1 \wedge \sigma^2, \quad (5b)$$

$$d\sigma^3 = \sigma^1 \wedge (\sigma^2 + \sigma^3). \quad (5c)$$

An orthonormal basis

$$ds^2 = \eta_{\mu\nu}\,\omega^\mu\omega^\nu , \quad (6)$$

where $\eta_{\mu\nu}$ is the Minkowski metric, is obtainable with the substitution

$$\omega^i = h_{ij}(t)\sigma^j , \quad (7)$$

with

$$g_{ij} = \sum_k h_{ki}h_{kj} . \quad (8)$$

The simplest choice which yields a nontrivial model is

$$\omega^1 = a\sigma^1, \quad (9a)$$

$$\omega^2 = b\sigma^2, \quad (9b)$$

$$\omega^3 = cf\sigma^2 + c\sigma^3 , \quad (9c)$$

where a, b, c, and f are the functions of t to be determined. A diagonal metric in synchronous form for either the vacuum case or perfect-fluid source does not exist, i.e., f cannot be set equal to zero and a consistent set of field equations obtained. This is seen most clearly in Eq. (19e). If f is set equal to zero, then c too must be zero.

Even with this rather special choice of metric the Einstein equations so obtained [Eqs. (19a)–(19f)] have so far proven to be intractable. However, the details of the equations are such as to suggest the possibility of substantial simplification if b is set equal to c. This choice is indeed made and a solution is readily obtained. Until Eqs. (19a)–(19f) are derived the more general form will be retained.

The inverses to Eqs. (7) are

$$\sigma^1 = \frac{1}{a}\omega^1 , \tag{10a}$$

$$\sigma^2 = \frac{1}{b}\omega^2 , \tag{10b}$$

$$\sigma^3 = -\frac{f}{b}\omega^2 + \frac{1}{c}\omega^3 . \tag{10c}$$

The exterior derivatives of the orthonormal basis one-forms are readily found by use of Eqs. (9) and substitution of Eqs. (5) and (10):

$$d\omega^0 = 0 , \tag{11a}$$

$$d\omega^1 = \frac{\dot a}{a}\omega^0 \wedge \omega^1 , \tag{11b}$$

$$d\omega^2 = \frac{\dot b}{b}\omega^0 \wedge \omega^2 + \frac{1}{a}\omega^1 \wedge \omega^2 , \tag{11c}$$

$$d\omega^3 = \frac{c\dot f}{b}\omega^0 \wedge \omega^2 + \frac{\dot c}{c}\omega^0 \wedge \omega^3 + \frac{c}{ab}\omega^1 \wedge \omega^2 + \frac{1}{a}\omega^1 \wedge \omega^3, \tag{11d}$$

where $\dot a \equiv da/dt$. Comparison of these equations with the relationship

$$d\omega^\alpha = -\tfrac{1}{2}C_{\beta\gamma}{}^\alpha \omega^\beta \wedge \omega^\gamma \tag{12}$$

provides, immediately, the structure coefficients $C_{\beta\gamma}{}^\alpha$. These satisfy the relationships

$$C_{\beta\gamma}{}^\alpha = -C_{\gamma\beta}{}^\alpha . \tag{13}$$

Define

$$C_{\beta\gamma\alpha} \equiv g_{\alpha\lambda} C_{\beta\gamma}{}^\lambda \tag{14a}$$

or, in the orthonormal frame,

$$C_{\beta\gamma\alpha} = \eta_{\alpha\lambda} C_{\beta\gamma}{}^\lambda . \tag{14b}$$

The nonvanishing coefficients then are

$$\begin{aligned} &C_{011} = -C_{101} = -\dot a/a, \quad C_{033} = -C_{303} = -\dot c/c , \\ &C_{022} = -C_{202} = -\dot b/b, \quad C_{123} = -C_{213} = -c/ab, \\ &C_{122} = -C_{212} = -1/a, \quad C_{133} = -C_{313} = -1/a, \\ &C_{023} = -C_{203} = -c\dot f/b. \end{aligned} \tag{15}$$

In the orthonormal frame the connection coefficients are

$$\Gamma_{\mu\beta\gamma} = \tfrac{1}{2}(C_{\mu\beta\gamma} + C_{\mu\gamma\beta} - C_{\beta\gamma\mu}). \tag{16}$$

Finally, the Ricci tensor is given by

$$R_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu} - \Gamma^\alpha{}_{\mu\beta}\Gamma^\beta{}_{\nu\alpha} + \Gamma^\alpha{}_{\beta\alpha}\Gamma^\beta{}_{\mu\nu} , \tag{17}$$

where

$$\Gamma^\alpha{}_{\beta\gamma} \equiv \eta^{\alpha\mu}\Gamma_{\mu\beta\gamma} .$$

In general, for the noncoordinate approach the commas appearing in Eq. (17) indicate the action of the basis vectors on the connection coefficients rather than partial differentiation. For spatially homogeneous space-times, however, such as treated here, it can be shown that the connection coefficients are functions solely of time and there is no inconsistency, inasmuch as the only operation which does not vanish is differentiation with respect to time.

Of the 10 components of the tensor, four vanish identically:

$$R_{02} \equiv R_{03} \equiv R_{12} \equiv R_{13} \equiv 0, \tag{18}$$

with the first two implying necessarily that

$$u_2 = u_3 = 0.$$

The field equations then are

$$\begin{aligned} R_{00} &= -\frac{d}{dt}\left(\frac{\dot a}{a} + \frac{\dot b}{b} + \frac{\dot c}{c}\right) - \left(\frac{\dot a}{a}\right)^2 - \left(\frac{\dot b}{b}\right)^2 - \left(\frac{\dot c}{c}\right)^2 - \frac{c^2\dot f^2}{2b^2} \\ &= (\rho + p)u_0 u_0 - \tfrac{1}{2}(\rho - p) - \Lambda , \end{aligned} \tag{19a}$$

$$\begin{aligned} R_{01} &= \frac{2\dot a}{a^2} - \frac{1}{a}\left(\frac{\dot b}{b} + \frac{\dot c}{c}\right) - \frac{c^2\dot f}{2ab^2} \\ &= (\rho + p)u_0 u_1 , \end{aligned} \tag{19b}$$

$$\begin{aligned} R_{11} &= \frac{d}{dt}\left(\frac{\dot a}{a}\right) + \frac{\dot a}{a}\left(\frac{\dot a}{a} + \frac{\dot b}{b} + \frac{\dot c}{c}\right) - \frac{2}{a^2} - \frac{c^2}{2a^2b^2} \\ &= (\rho + p)u_1 u_1 + \tfrac{1}{2}(\rho - p) + \Lambda , \end{aligned} \tag{19c}$$

$$\begin{aligned} R_{22} &= \frac{d}{dt}\left(\frac{\dot b}{b}\right) + \frac{\dot b}{b}\left(\frac{\dot a}{a} + \frac{\dot b}{b} + \frac{\dot c}{c}\right) + \frac{c^2\dot f^2}{2b^2} - \frac{c^2}{2a^2b^2} - \frac{2}{a^2} \\ &= \tfrac{1}{2}(\rho - p) + \Lambda , \end{aligned} \tag{19d}$$

$$\begin{aligned} R_{23} &= \frac{d}{dt}\left(\frac{c\dot f}{2b}\right) + \frac{c\dot f}{2b}\left(\frac{\dot a}{a} + 2\frac{\dot c}{c}\right) - \frac{c}{a^2b} \\ &= 0 , \end{aligned} \tag{19e}$$

$$R_{33}=\frac{d}{dt}\left(\frac{\dot c}{c}\right)+\frac{\dot c}{c}\left(\frac{\dot a}{a}+\frac{\dot b}{b}+\frac{\dot c}{c}\right)-\frac{c^2\dot f^2}{2b^2}+\frac{c^2}{2a^2b^2}-\frac{2}{a^2}$$
$$=\tfrac{1}{2}(\rho-p)+\Lambda\,. \qquad (19f)$$

THE METRIC

Equation (19e) implies that unless the off-diagonal term f is retained, no solution will exist. A solution may be obtained most simply if b is set equal to c. Then

$$R_{22}-R_{33}=\dot f^2-1/a^2=0 \qquad (20)$$

and

$$\dot f=\pm 1/a. \qquad (21)$$

Substitution of this into Eq. (19e) yields

$$\dot b/b=\pm 1/a. \qquad (22)$$

Resubstitution of these two expressions into either Eq. (19d) or Eq. (19f) yields

$$\tfrac{1}{2}(\rho-p)+\Lambda=0. \qquad (23)$$

The remaining field equations are thus reduced to

$$R_{00}=-\frac{d}{dt}\left(\frac{\dot a}{a}\right)\pm 2\frac{\dot a}{a^2}-\left(\frac{\dot a}{a}\right)^2-\frac{5}{2a^2}$$
$$=(\rho+p)u_0u_0, \qquad (24a)$$

$$R_{01}=\mp 2\frac{\dot a}{a^2}+\frac{5}{2a^2}$$
$$=(\rho+p)u_0u_1, \qquad (24b)$$

$$R_{11}=\frac{d}{dt}\left(\frac{\dot a}{a}\right)+\left(\frac{\dot a}{a}\right)^2\pm 2\frac{\dot a}{a^2}-\frac{5}{2a^2}$$
$$=(\rho+p)u_1u_1\,. \qquad (24c)$$

Then

$$R_{00}+2R_{01}+R_{11}=(\rho+p)(u_0u_0+2u_0u_1+u_1u_1)$$
$$=0, \qquad (25)$$

with the implication that either the density or pressure are negative—an untenable hypothesis—or that $u_0=-u_1$. This latter condition is inconsistent with the constraint $\eta^{\mu\nu}u_\mu u_\nu=-1$. The only consistent set of values is $\rho=p=u_i=0$. Then, necessarily, $\Lambda=0$. The model, with the particular metric, i.e., $b=c$, $f\neq 0$, simply does not admit other than a vacuum solution with zero cosmological constant. (See, however, comments below.)

This solution is readily found. $R_{01}=0$ yields

$$a=\pm\tfrac{5}{4}t+k\,, \qquad (26)$$

with the concomitant

$$b=b_0(\pm\tfrac{5}{4}t+k)^{4/5} \qquad (27)$$

and

$$f=\ln f_0(\pm\tfrac{5}{4}t+k)^{4/5}, \qquad (28)$$

where k, b_0, and f_0 are constants of integration. The latter two are scale factors and may be set equal to 1. The constant k sets the time of the singularity. It may be chosen arbitrarily; the most convenient choice is $k=0$.

The choice of sign for t manifested in Eqs. (26), (27), and (28) corresponds to describing the solution in either the positive or negative time domain. It is convenient to take $t>0$.

The metric is thus

$$ds^2=-\sigma^0\sigma^0+(\tfrac{5}{4}t)^2\sigma^1\sigma^1+(\tfrac{5}{4}t)^{8/5}\{[1+\ln^2(\tfrac{5}{4}t)^{4/5}]\sigma^2\sigma^2+\ln(\tfrac{5}{4}t)^{4/5}(\sigma^2\sigma^3+\sigma^3\sigma^2)+\sigma^3\sigma^3\}\,. \qquad (29)$$

The point at $t=0$ is more than just a coordinate singularity. It is a true singularity. Indeed, the nonvanishing components of the Riemann-Christoffel curvature tensor all diverge like $1/t^2$ as $t\to 0$; the nonvanishing curvature invariants diverge like $1/t^4$ or $1/t^6$. The situation is quite different as $t\to\infty$. Here, all components of the curvature tensor vanish. The singularity at $t=\infty$ is thus a coordinate singularity. (There may exist here a condition of geodesic incompleteness, and this matter is currently being explored.)

The evolution of this model is the same as that of the "big bang" models except for the remarkable feature that it is devoid of matter.

COMMENTS

Among the important features of the model is its rejection of matter, at least in the form of a perfect fluid. One has no choice, and it would be of considerable interest to understand why this transpires. An investigation of this point is currently being pursued. It is hoped to determine if a more general metric or more general source, e.g., a charged dust, might survive. In this connection it should be remembered that the requirement of spatial homogeneity in Bianchi models implies the nonadmissibility of fields which vary spatially. Thus, radiation as a source is ruled out except in lowest mode.

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[1]G. F. R. Ellis and M. A. H. MacCallum, Commun. Math. Phys. 12, 108 (1969).

[2]M. P. Ryan, Jr. and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton Univ. Press, Princeton, New Jersey, 1975).

[3]M. A. H. MacCallum, *Cargèse Lectures in Physics* (Gordon and Breach, New York, 1973), Vol. 6, pp. 61–174.

[4]There is a school of thought, however [see, e.g., D. H. Brill, Phys. Rev. 133, B845 (1964)], which believes that such curvature is not essentially anti-Machian, but is a manifestation of *gravitational radiation* which is present in lowest-possible mode. Such a field could exist without destroying the homogeneity. The question is still open.

[5]A. H. Taub, Ann. Math. 53, 472 (1951).

[6]P. J. E. Peebles, *Physical Cosmology* (Princeton Univ. Press, Princeton, New Jersey, 1971).